ALGEBRAIC INTEGERS AS SPECIAL VALUES OF MODULAR UNITS

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Abstract. Let $\varphi(\tau)=\eta((\tau+1)/2)^2/\sqrt{2\pi}e^{\frac{\pi i}{4}}\eta(\tau+1)$ where $\eta(\tau)$ is the Dedekind eta-function. We show that if τ_0 is an imaginary quadratic number with $\mathrm{Im}(\tau_0)>0$ and m is an odd integer, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$. This is a generalization of Theorem 4.4 given in [1]. On the other hand, let K be an imaginary quadratic field and θ_K be an element of K with $\mathrm{Im}(\theta_K)>0$ which generators the ring of integers of K over $\mathbb{Z}$. We develop a sufficient condition of m for $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ to become a unit.

1. Introduction

The *Dedekind eta-function* $\eta(\tau)$ is defined to be the following infinite product expansion

$$\eta(\tau)=\sqrt{2\pi}e^{\frac{\pi i}{4}}q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\quad(\tau\in\mathfrak{H})\tag{1.1}$$

where $q=e^{2\pi i\tau}$ and $\mathfrak{H}=\{\tau\in\mathbb{C}:\mathrm{Im}(\tau)>0\}$. Define a function

$$\varphi(\tau)=\frac{1}{\sqrt{2\pi}e^{\frac{\pi i}{4}}}\frac{\eta((\tau+1)/2)^2}{\eta(\tau+1)}=\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})^2(1-q^n)\quad(\tau\in\mathfrak{H}).\tag{1.2}$$

Motivated by Ramanujan's evaluation of $\varphi(mi)/\varphi(i)$ for some positive integers m ([8]), which are algebraic numbers, Berndt-Chan-Zhang proved the following theorem.

Theorem 1.1 ([1] Theorem 4.4)**.** *Let m and n be positive integers. If m is odd, then $\sqrt{2m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $2\sqrt{m}$, while if m is even, then $2\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

In this paper we shall revisit the above theorem and improve the statement when m is odd, as follows:

Theorem 1.2. *Let m be a positive integer and $\tau_0\in\mathfrak{H}$ be imaginary quadratic. Then $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$. In particular, if m is odd, then $\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$.*

For $(r_1,r_2)\in\mathbb{Q}^2-\mathbb{Z}^2$, the *Siegel function* $g_{(r_1,r_2)}(\tau)$ is defined by

$$g_{(r_1,r_2)}(\tau)=-q^{\frac{1}{2}\mathbf{B}_2(r_1)}e^{\pi i r_2(r_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q^nq_z)(1-q^nq_z^{-1})\qquad(\tau\in\mathfrak{H})\tag{1.3}$$

where $\mathbf{B}_2(x)=x^2-x+1/6$ is the second Bernoulli polynomial and $q_z=e^{2\pi iz}$ with $z=r_1\tau+r_2$. We shall first express the function $\varphi(m\tau)/\varphi(\tau)$ as a product of certain eta-quotient and Siegel functions (Proposition 2.6(i)). Then, we shall prove Theorem 1.2 in §3 by using integrality of Siegel functions over $\mathbb{Z}[j(\tau)]$ (Proposition 2.2) where

$$j(\tau)=\left(\frac{\eta(\tau)^{24}+2^8\eta(2\tau)^{24}}{\eta(\tau)^{16}\eta(2\tau)^8}\right)^3=q^{-1}+744+196884q+21493760q^2+\cdots$$

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is the well-known *modular j-function* ([2] Theorem 12.17).

On the other hand, let K be an imaginary quadratic field with discriminant d_K, and define

$$\theta_K = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{for } d_K \equiv 0 \pmod 4 \\ \frac{-1+\sqrt{d_K}}{2} & \text{for } d_K \equiv 1 \pmod 4, \end{cases} \tag{1.4}$$

which generates the ring of integers of K over $\mathbb{Z}$. Ramachandra showed that if N (≥ 2) is an integer with more than one prime ideal factor K, then $g_{(0,\frac{1}{N})}(\theta_K)^{12N}$ is a unit (Proposition 4.5). This fact, together with the Shimura's reciprocity law (Proposition 4.7), will enable us to prove the following theorem in §4.

Theorem 1.3. *If m (≥ 3) is an odd integer whose prime factors split in K, then $\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K)$ is a unit.*

2. Arithmetic properties of Siegel functions

In this section we shall examine some arithmetic properties of Siegel functions. For the classical theory of modular functions, one can refer to [6] or [9].

For each positive integer N, let $\zeta_N = e^{\frac{2\pi i}{N}}$ and $\mathcal{F}_N$ be the field of meromorphic modular functions of level N whose Fourier coefficients belong to the N^{th} cyclotomic field $\mathbb{Q}(\zeta_N)$.

Proposition 2.1. *For each positive integer N, $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ whose Galois group is isomorphic to*

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} = G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$$

where

$$G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \ : \ d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

Here, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in G_N$ acts on $\sum_{n=-\infty}^{\infty} c_n q^{\frac{n}{N}} \in \mathcal{F}_N$ by

$$\sum_{n=-\infty}^{\infty} c_n q^{\frac{n}{N}} \mapsto \sum_{n=-\infty}^{\infty} c_n^{\sigma_d} q^{\frac{n}{N}}$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ induced by $\zeta_N \mapsto \zeta_N^d$. And, for an element $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ let $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ be a preimage of γ via the natural surjection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$. Then γ acts on $h \in \mathcal{F}_N$ by composition

$$h \mapsto h \circ \gamma'$$

as linear fractional transformation.

Proof. See [6] Chapter 6 Theorem 3. □

Proposition 2.2. *Let $(r_1, r_2) \in \frac{1}{N}\mathbb{Z}^2 - \mathbb{Z}^2$ for some integer $N \geq 2$.*

(i) *$g_{(r_1,r_2)}(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$. Namely, $g_{(r_1,r_2)}(\tau)$ is a zero of a monic polynomial whose coefficients are in $\mathbb{Z}[j(\tau)]$.*

(ii) *Suppose that (r_1, r_2) has the primitive denominator N (that is, N is the smallest positive integer such that $(Nr_1, Nr_2) \in \mathbb{Z}^2$). If N is composite (that is, N has at least two prime factors), then $g_{(r_1,r_2)}(\tau)^{-1}$ is also integral over $\mathbb{Z}[j(\tau)]$.*

(iii) *$g_{(r_1,r_2)}(\tau)$ is holomorphic and has no zeros and poles on $\mathfrak{H}$. Furthermore, $g_{(r_1,r_2)}(\tau)$ (respectively, $g_{(r_1,r_2)}(\tau)^{12N/\gcd(6,N)}$) belongs to $\mathcal{F}_{12N^2}$ (respectively, $\mathcal{F}_N$).*

Proof. See [4] §3, [5] Chapter 2 Theorems 2.2, 1.2 and Chapter 3 Theorem 5.2. □

Remark 2.3. Let $g(\tau)$ be an element of $\mathcal{F}_N$ for some positive integer N. If both $g(\tau)$ and $g(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$ (respectively, $\mathbb{Z}[j(\tau)]$), then $g(\tau)$ is called a *modular unit* (respectively, *modular unit over* $\mathbb{Z}$) of level N. As is well-known, $g(\tau)$ is a modular unit if and only if it has no zeros and poles on $\mathfrak{H}$ ([5] Chapter 2 §2 or [4] §2). Hence $g_{(r_1,r_2)}(\tau)$ is a modular unit for any $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$ by (iii). Moreover, if (r_1, r_2) has the composite primitive denominator, then $g_{(r_1,r_2)}(\tau)$ is a modular unit over $\mathbb{Z}$ by (ii).

We recall some basic transformation formulas of Siegel functions.

Proposition 2.4. *Let $r = (r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$.*

(i) *We have*

$$g_{-r}(\tau) = g_{(-r_1,-r_2)}(\tau) = -g_r(\tau).$$

(ii) *For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we get*

$$\begin{array}{rcccl} g_r(\tau) \circ S & = & \zeta_{12}^9 g_{rS}(\tau) & = & \zeta_{12}^9 g_{(r_2,-r_1)}(\tau) \\ g_r(\tau) \circ T & = & \zeta_{12} g_{rT}(\tau) & = & \zeta_{12} g_{(r_1,r_1+r_2)}(\tau). \end{array}$$

Hence we obtain that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$g_r(\tau) \circ \gamma = \varepsilon g_{r\gamma}(\tau)$$

with ε a 12^{th} root of unity (depending on γ).

(iii) *For $s = (s_1, s_2) \in \mathbb{Z}^2$ we have*

$$g_{r+s}(\tau) = g_{(r_1+s_1,r_2+s_2)}(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 r_2 - s_2 r_1)} g_r(\tau).$$

(iv) *Let $r \in \frac{1}{N}\mathbb{Z}^2 - \mathbb{Z}^2$ for some integer $N \geq 2$. Each element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ acts on $g_r(\tau)^{12N/\gcd(6,N)}$ by*

$$\left(g_r(\tau)^{\frac{12N}{\gcd(6,N)}} \right)^{\alpha} = g_{r\alpha}(\tau)^{\frac{12N}{\gcd(6,N)}} = g_{(r_1 a + r_2 c, r_1 b + r_2 d)}(\tau)^{\frac{12N}{\gcd(6,N)}}.$$

(v) *We have the order formula*

$$\mathrm{ord}_q g_r(\tau) = \frac{1}{2}\mathbf{B}_2(\langle r_1 \rangle)$$

where $\langle r_1 \rangle$ is the fractional part of r_1 in the interval $[0, 1)$.

Proof. See [4] Propositions 2.4, 2.5 and [5] p. 31. □

Remark 2.5. The expression $r\alpha$ in (iv) is well-defined by (i) and (iii).

Proposition 2.6. (i) *We can express $\varphi(\tau)$ as*

$$\varphi(\tau) = -\frac{1}{\sqrt{2\pi}} \eta(\tau) g_{(\frac{1}{2},\frac{1}{2})}(\tau).$$

(ii) *We dervie*

$$g_{(0,\frac{1}{2})}(\tau) g_{(\frac{1}{2},0)}(\tau) g_{(\frac{1}{2},\frac{1}{2})}(\tau) = 2e^{\frac{\pi i}{4}}.$$

(iii) *If m (≥ 3) is an odd integer, then we have the relation*

$$\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau)} = (-1)^{\frac{m-1}{2}} \prod_{k=1}^{m-1} g_{(\frac{1}{2},\frac{1}{2}+\frac{k}{m})}(\tau).$$

Proof. (i) By the definition (1.3) we have

$$g_{(\frac{1}{2},\frac{1}{2})}(\tau) = -q^{\frac{1}{2}\mathbf{B}_2(\frac{1}{2})} e^{-\frac{\pi i}{4}} (1 + q^{\frac{1}{2}}) \prod_{n=1}^{\infty} (1 + q^{n+\frac{1}{2}})(1 + q^{n-\frac{1}{2}}) = -e^{-\frac{\pi i}{4}} q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^2.$$

One can readily obtain the assertion by the definition (1.1) of $\eta(\tau)$ and the infinite product expansion (1.2) of $\varphi(\tau)$.

(ii) Put $g(\tau) = g_{(0,\frac{1}{2})}(\tau)g_{(\frac{1}{2},0)}(\tau)g_{(\frac{1}{2},\frac{1}{2})}(\tau)$, which is an element of $\mathcal{F}_{48}$ by Proposition 2.2(iii). For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we derive that

$$\begin{aligned}
\mathrm{ord}_q(g(\tau)\circ\alpha) &= \mathrm{ord}_q\left(g_{(\frac{c}{2},\frac{d}{2})}(\tau)g_{(\frac{a}{2},\frac{b}{2})}(\tau)g_{(\frac{a+c}{2},\frac{b+d}{2})}(\tau)\right) \text{ by Proposition 2.4(ii)} \\
&= \frac{1}{2}\mathbf{B}_2\left(\left\langle\frac{c}{2}\right\rangle\right) + \frac{1}{2}\mathbf{B}_2\left(\left\langle\frac{a}{2}\right\rangle\right) + \frac{1}{2}\mathbf{B}_2\left(\left\langle\frac{a+c}{2}\right\rangle\right) \text{ by Proposition 2.4(v)} \\
&= \frac{1}{2}\mathbf{B}_2(0) + 2\cdot\frac{1}{2}\mathbf{B}_2\left(\frac{1}{2}\right) \text{ because both } a \text{ and } c \text{ cannot be even} \\
&= 0.
\end{aligned}$$

This observation implies that $g(\tau)$ is holomorphic at every cusp. Thus $g(\tau)$ is a holomorphic function on the modular curve of level 48 (which is a compact Riemann surface, or an algebraic curve); and hence it must be a constant. It follows that

$$\begin{aligned}
g(\tau) &= -2e^{-\frac{3\pi i}{4}}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{n-\frac{1}{2}})^2(1+q^{n-\frac{1}{2}})^2 \text{ by the definition (1.3)} \\
&= -2e^{-\frac{3\pi i}{4}}\lim_{q\to 0}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^{n-\frac{1}{2}})^2(1+q^{n-\frac{1}{2}})^2 \\
&= 2e^{\frac{\pi i}{4}}.
\end{aligned}$$

(iii) By the definition (1.3) we have

$$\begin{aligned}
\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau)} &= \frac{-q^{\frac{m}{2}\mathbf{B}_2(\frac{1}{2})}e^{-\frac{\pi i}{4}}(1+q^{\frac{m}{2}})\prod_{n=1}^{\infty}(1+q^{mn+\frac{m}{2}})(1+q^{mn-\frac{m}{2}})}{-q^{\frac{1}{2}\mathbf{B}_2(\frac{1}{2})}e^{-\frac{\pi i}{4}}(1+q^{\frac{1}{2}})\prod_{n=1}^{\infty}(1+q^{n+\frac{1}{2}})(1+q^{n-\frac{1}{2}})} \\
&= q^{\frac{1-m}{24}}\prod_{n=1}^{\infty}\left(\frac{1+q^{m(n-\frac{1}{2})}}{1+q^{n-\frac{1}{2}}}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{k=1}^{m-1} g_{(\frac{1}{2},\frac{1}{2}+\frac{k}{m})}(\tau) \\
&= \prod_{k=1}^{m-1}\left(-q^{\frac{1}{2}\mathbf{B}_2(\frac{1}{2})}e^{\pi i(\frac{1}{2}+\frac{k}{m})(-\frac{1}{2})}(1+q^{\frac{1}{2}}\zeta_m^k)\prod_{n=1}^{\infty}(1+q^{n+\frac{1}{2}}\zeta_m^k)(1+q^{n-\frac{1}{2}}\zeta_m^{-k})\right) \\
&= (-1)^{m-1}e^{\pi i\frac{1-m}{2}}q^{\frac{1-m}{24}}\prod_{k=1}^{m-1}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}\zeta_m^k)(1+q^{n-\frac{1}{2}}\zeta_m^{-k}) \\
&= (-1)^{\frac{1-m}{2}}q^{\frac{1-m}{24}}\prod_{n=1}^{\infty}\prod_{k=1}^{m-1}(1+q^{n-\frac{1}{2}}\zeta_m^k)^2 \text{ because } m \text{ is odd} \\
&= (-1)^{\frac{1-m}{2}}q^{\frac{1-m}{24}}\prod_{n=1}^{\infty}\left(\frac{1+q^{m(n-\frac{1}{2})}}{1+q^{n-\frac{1}{2}}}\right)^2 \text{ by the identity } \frac{1+X^m}{1+X} = \frac{1-(-X)^m}{1-(-X)} = \prod_{k=1}^{m-1}(1-(-X)\zeta_m^k).
\end{aligned}$$

This proves (iii). □

3. Proof of Theorem 1.2

Let

$$\Delta(\tau) = \eta(\tau)^{24} = (2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24} \quad (\tau\in\mathfrak{H}) \tag{3.1}$$

be the *modular discriminant function.*

Proposition 3.1. *Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

(i) *$j(\tau_0)$ is an algebraic integer.*

(ii) *Let a, b and d be integers with $ad > 0$ and $\gcd(a, b, d) = 1$. Then, $a^{12}\Delta((a\tau_0 + b)/d)/\Delta(\tau_0)$ is an algebraic integer dividing $(ad)^{12}$.*

Proof. See [6] Chapter 5 Theorem 4 and Chapter 12 Theorem 4. □

Proposition 3.2. *Let m be a positive integer and $\tau_0 \in \mathfrak{H}$ be imaginary quadratic.*

(i) *$\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$.*

(ii) *$2g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)/g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)$ is an algebraic integer dividing 4. In particular, if m is odd, then $g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)/g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)$ is a unit.*

Proof. (i) Applying Proposition 3.1(ii) with $(a, b, d) = (m, 0, 1)$, wee see that

$$m^{12}\frac{\Delta(m\tau_0)}{\Delta(\tau_0)} = \left(\sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)}\right)^{24}$$

is an algebraic integer dividing m^{12}. We get the assertion by taking 24^{th} root.

(ii) We obtain from Proposition 2.6(ii) that

$$\begin{aligned} 2\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)} &= e^{-\frac{\pi i}{4}} g_{(0,\frac{1}{2})}(\tau_0) g_{(\frac{1}{2},0)}(\tau_0) g_{(\frac{1}{2},\frac{1}{2})}(\tau_0) \frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)} \\ &= e^{-\frac{\pi i}{4}} \left(g_{(0,\frac{1}{2})}(\tau_0) g_{(\frac{1}{2},0)}(\tau_0)\right) g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0). \end{aligned}$$

By Propositions 2.2(i) and 3.1(i), the values $g_{(0,\frac{1}{2})}(\tau_0)g_{(\frac{1}{2},0)}(\tau_0)$, $g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)$, $g_{(0,\frac{1}{2})}(m\tau_0)g_{(\frac{1}{2},0)}(m\tau_0)$ and $g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)$ are algebraic integers. Moreover, since

$$\left(g_{(0,\frac{1}{2})}(\tau_0) g_{(\frac{1}{2},0)}(\tau_0)\right) g_{(\frac{1}{2},\frac{1}{2})}(\tau_0) = \left(g_{(0,\frac{1}{2})}(m\tau_0) g_{(\frac{1}{2},0)}(m\tau_0)\right) g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0) = 2e^{\frac{\pi i}{4}}$$

by Proposition 2.6(ii), both $g_{(0,\frac{1}{2})}(\tau_0)g_{(\frac{1}{2},0)}(\tau_0)$ and $g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)$ are algebraic integers dividing 2. Hence $2g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)/g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)$ is an algebraic integer dividing $2 \cdot 2 = 4$.

Now, suppose that m (≥ 3) is odd. Recall the relation

$$\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau)} = (-1)^{\frac{m-1}{2}} \prod_{k=1}^{m-1} g_{(\frac{1}{2},\frac{1}{2}+\frac{k}{m})}(\tau)$$

given in Proposition 2.6(iii). Since each vector $(\frac{1}{2}, \frac{1}{2} + \frac{k}{m})$ has the composite primitive denominator, $g_{(\frac{1}{2},\frac{1}{2}+\frac{k}{m})}(\tau)$ is a modular unit over $\mathbb{Z}$ by Proposition 2.2(ii); and hence so is $g_{(\frac{1}{2},\frac{1}{2})}(m\tau)/g_{(\frac{1}{2},\frac{1}{2})}(\tau)$. Therefore, $g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)/g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)$ is a unit by Proposition 3.1(i). □

We are ready to prove Theorem 1.2. Let m be a positive integer and $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. By Proposition 2.6(i) we can express

$$2\sqrt{m}\frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot 2\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)}.$$

Hence $2\sqrt{m}\varphi(m\tau_0)/\varphi(\tau_0)$ is an algebraic integer dividing $4\sqrt{m}$ by Proposition 3.2(i) and (ii). Similarly, if m is odd, then

$$\sqrt{m}\frac{\varphi(m\tau_0)}{\varphi(\tau_0)} = \sqrt{m}\frac{\eta(m\tau_0)}{\eta(\tau_0)} \cdot \frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau_0)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau_0)} \tag{3.2}$$

is an algebraic integer dividing $\sqrt{m}$. This completes the proof of Theorem 1.2.

Now, we revisit and improve Theorem 1.1 as a corollary.

Corollary 3.3. *Let m and n be positive integers. If m is odd, then $\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $\sqrt{m}$, while if m is even, then $2\sqrt{m}\varphi(mni)/\varphi(ni)$ is an algebraic integer dividing $4\sqrt{m}$.*

Proof. We get the assertion by setting $\tau_0 = ni$ in Theorem 1.2. □

Remark 3.4. Berndt-Chan-Zhang used only Proposition 3.1(ii) in order to achieve Theorem 1.1.

4. Proof of Theorem 1.3

Proposition 4.1. *Let m (≥ 2) be an integer.*

(i) *We have the relation*

$$\prod_{\substack{a,b\in\mathbb{Z}\\ 0\leq a,b<m,\ (a,b)\neq(0,0)}} g_{(\frac{a}{m},\frac{b}{m})}(\tau)^{12m} = m^{12m}.$$

(ii) *We derive*

$$\prod_{k=1}^{m-1} g_{(0,\frac{k}{m})}(\tau) = i^{m-1}\left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2.$$

Proof. (i) See [5] p. 45 Example.
(ii) We deduce that

$$\begin{aligned}
\prod_{k=1}^{m-1} g_{(0,\ \frac{k}{m})}(\tau) &= \prod_{k=1}^{m-1}\left(-q^{\frac{1}{2}\mathbf{B}_2(0)}\zeta_{2m}^{-k}(1-\zeta_m^k)\prod_{n=1}^{\infty}(1-q^n\zeta_m^k)(1-q^n\zeta_m^{-k})\right) \text{ by the definition (1.3)}\\
&= i^{m-1}mq^{\frac{m-1}{12}}\prod_{n=1}^{\infty}\left(\frac{1-q^{mn}}{1-q^n}\right)^2\\
&\quad \text{by the identity } \frac{1-X^m}{1-X} = 1+X+\cdots+X^{m-1} = \prod_{k=1}^{m-1}(1-X\zeta_m^k)\\
&= i^{m-1}\left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2 \text{ by the definition (1.1).}
\end{aligned}$$

□

Remark 4.2. Let $\tau_0 \in \mathfrak{H}$ be imaginary quadratic. By Propositions 2.2(i), 3.1(i) and 4.1(i), $\prod_{k=1}^{m-1} g_{(0,\frac{k}{m})}(\tau_0)$ is an algebraic integer dividing m. It follows from Proposition 4.1(ii) that $\sqrt{m}\eta(m\tau_0)/\eta(\tau_0)$ is an algebraic integer dividing $\sqrt{m}$. This gives another proof of Proposition 3.2(i).

From now on, we let K be an imaginary quadratic field and θ_K be as in (1.4). We denote H_K and $K_{(N)}$ the Hilbert class field and the ray class field modulo N (≥ 1) of K, respectively.

Proposition 4.3 (Main theorem of complex multiplication)**.** *We have*

$$K_{(N)} = K\mathcal{F}_N(\theta_K) = K\Big(h(\theta_K)\ :\ h\in\mathcal{F}_N \text{ is defined and finite at } \theta_K\Big).$$

Proof. See [6] Chapter 10 Corollary to Theorem 2 or [9] Chapter 6. □

Corollary 4.4. *If m (≥ 3) is an odd integer, then $(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$ lies in $K_{(48m^2)}$.*

Proof. We see that

$$\begin{aligned}
\left(\sqrt{m}\frac{\varphi(m\tau)}{\varphi(\tau)}\right)^2 &= \left(\sqrt{m}\frac{\eta(m\tau)}{\eta(\tau)}\right)^2\left(\frac{g_{(\frac{1}{2},\frac{1}{2})}(m\tau)}{g_{(\frac{1}{2},\frac{1}{2})}(\tau)}\right)^2 \text{ by Proposition 2.6(i)}\\
&= (-1)^{\frac{1-m}{2}}\prod_{k=1}^{m-1} g_{(0,\frac{k}{m})}(\tau)g_{(\frac{1}{2},\frac{1}{2}+\frac{k}{m})}(\tau)^2 \text{ by Propositions 4.1(ii) and 2.6(iii).} \qquad (4.1)
\end{aligned}$$

Hence $(\sqrt{m}\varphi(m\tau)/\varphi(\tau))^2$ belongs to $\mathcal{F}_{48m^2}$ by Proposition 2.2(iii). Therefore, $(\sqrt{m}\varphi(m\theta_K)/\varphi(\theta_K))^2$ lies in $K_{(48m^2)}$ by Proposition 4.3. □

Proposition 4.5. *If N (≥ 2) is an integer with more than one prime ideal factor in K, then $g_{(0,\frac{1}{N})}(\theta_K)^{12N}$ is a unit which lies in $K_{(N)}$.*

Proof. See [7] §6. □

Remark 4.6. In [3] authors proved that if K is an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}\sqrt{-3})$, then $g_{(0,\frac{1}{N})}(\theta_K)^{12N}$ is a primitive generator of $K_{(N)}$ over K, which is called a *Siegel-Ramachandra invariant* ([5] Chapter 11 §1 or [7]).

On the other hand, we have the following explicit description of the Shimura's reciprocity law which connects the class field theory with the theory of modular functions, due to Stevenhagen.

Proposition 4.7 (Shimura's reciprocity law)**.** *Let $\min(\theta_K,\ \mathbb{Q}) = X^2 + BX + C \in \mathbb{Z}[X]$. For every positive integer N the matrix group*

$$W_{K,N} = \left\{ \begin{pmatrix} t-Bs & -Cs \\ s & t \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \ : \ t,\ s \in \mathbb{Z}/N\mathbb{Z} \right\}$$

gives rise to the surjection

$$\begin{array}{ccc} W_{K,N} & \longrightarrow & \mathrm{Gal}(K_{(N)}/H_K) \\ \alpha & \mapsto & \Big(h(\theta) \mapsto h^{\alpha}(\theta_K)\Big) \end{array} \tag{4.2}$$

where $h \in \mathcal{F}_N$ is defined and finite at θ_K. Its kernel is given by

$$\begin{cases} \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}) \\ \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},\ \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \\ \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. See [10] §3. □

Proposition 4.8. *If m (≥ 2) is an integer whose prime factors split in K, then $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ is a unit.*

Proof. We get from Proposition 4.1(ii) that

$$\left(\sqrt{m}\frac{\eta(m\theta_K)}{\eta(\theta_K)}\right)^{24m} = \prod_{k=1}^{m-1} g_{(0,\frac{k}{m})}(\theta_K)^{12m}. \tag{4.4}$$

For each $1 \leq k \leq m-1$, let us write

$$\frac{k}{m} = \frac{a}{b} \text{ with positive integers } a \text{ and } b \text{ such that } \gcd(a,b) = 1.$$

Since $g_{(0,\frac{1}{b})}(\theta_K)^{12b}$ lies in $K_{(b)}$ by Propositions 2.2(iii) and 4.3, and $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in W_{K,b} \simeq \mathrm{Gal}(K_{(b)}/H_K)$, we derive that

$$\begin{aligned} \left(g_{(0,\frac{1}{b})}(\theta_K)^{12b}\right)^{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)} &= \left(g_{(0,\frac{1}{b})}(\tau)^{12b}\right)^{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)}(\theta_K) \text{ by Proposition 4.7} \\ &= \left(g_{(0,\frac{1}{b})\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)}(\tau)^{12b}\right)(\theta_K) \text{ by Proposition 2.4(iv)} \\ &= g_{(0,\frac{a}{b})}(\theta_K)^{12b}. \end{aligned}$$

On the other hand, since b has more than one prime ideal factor in K by the assumption on m, $g_{(0,\frac{1}{b})}(\theta_K)^{12b}$ is a unit by Proposition 4.5. Hence $g_{(0,\frac{k}{m})}(\theta_K)^{12m} = (g_{(0,\frac{a}{b})}(\theta_K)^{12b})^{m/b}$ is also a unit. Therefore $\sqrt{m}\eta(m\theta_K)\eta(\theta_K)$ becomes a unit by the relation (4.4). $\square$

Now, we can prove Theorem 1.3. Let m (≥ 3) be an odd integer whose prime factors split in K. Since both $\sqrt{m}\eta(m\theta_K)/\eta(\theta_K)$ and $g_{(\frac{1}{2},\frac{1}{2})}(m\theta_K)/g_{(\frac{1}{2},\frac{1}{2})}(\theta_K)$ are units by Propositions 4.8 and 3.2(ii), the result follows from the expression (3.2) with $\tau_0 = \theta_K$. This completes the proof.

Corollary 4.9. *Let m (≥ 3) be an odd integer whose prime factors p satisfy $p \equiv 1 \pmod 4$. Then $\sqrt{m}\varphi(mi)/\varphi(i)$ is a unit.*

Proof. If $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = i$. For each prime factor p of m, the fact $p \equiv 1 \pmod 4$ implies that p splits in K ([2] Corollary 5.17). We get the assertion by applying Theorem 1.3. $\square$

We close this section by evaluating $\sqrt{m}\varphi(mi)/\varphi(i)$ for $m = 3$ and 5, explicitly.

Example 4.10. We shall evaluate $\sqrt{3}\varphi(3i)/\varphi(i)$. If $K = \mathbb{Q}(\sqrt{-1})$, then $\theta_K = i$ and $H_K = K$ ([2] Theorem 12.34). By Proposition 4.7 we have

$$\begin{aligned}\mathrm{Gal}(K_{(6)}/K) &\simeq W_{K,6}/\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \pm\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\}\\ &= \left\{\alpha_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \alpha_2=\begin{pmatrix}1&-2\\2&1\end{pmatrix},\ \alpha_3=\begin{pmatrix}1&-4\\4&1\end{pmatrix},\ \alpha_4=\begin{pmatrix}3&-2\\2&3\end{pmatrix}\right\}.\end{aligned}$$

Since

$$\begin{aligned} x = \left(\sqrt{3}\frac{\varphi(3i)}{\varphi(i)}\right)^{24} &= g_{(0,\frac{1}{3})}(i)^{12}g_{(0,\frac{2}{3})}(i)^{12}g_{(\frac{1}{2},\frac{5}{6})}(i)^{24}g_{(\frac{1}{2},\frac{7}{6})}(i)^{24} \text{ by (4.1)}\\ &= g_{(0,\frac{1}{3})}(i)^{24}g_{(\frac{1}{2},\frac{1}{6})}(i)^{48} \text{ by Proposition 2.4(i) and (iii)}\\ &\approx 72954,\end{aligned}$$

x lies in $K_{(6)}$ by Propositions 2.2(iii) and 4.3. Hence its conjugates $x_k = x^{\alpha_k}$ $(1 \leq k \leq 4)$ over K are

$$\begin{aligned} x_1 &= g_{(0,\frac{1}{3})}(i)^{24}g_{(\frac{1}{2},\frac{1}{6})}(i)^{48},\\ x_2 &= g_{(\frac{2}{3},\frac{1}{3})}(i)^{24}g_{(\frac{5}{6},\frac{1}{6})}(i)^{48},\\ x_3 &= g_{(\frac{1}{3},\frac{1}{3})}(i)^{24}g_{(\frac{1}{6},\frac{1}{6})}(i)^{48},\\ x_4 &= g_{(\frac{2}{3},0)}(i)^{24}g_{(\frac{5}{6},\frac{1}{2})}(i)^{48}\end{aligned}$$

with some multiplicity by Propositions 4.7 and 2.4(iv). We claim that the minimal polynomial of x over K has integeral coefficients. Indeed, since x is a real algebraic integer by the definition (1.2) and Theorem 1.2, we have

$$[\mathbb{Q}(x):\mathbb{Q}] = \frac{[K(x):K]\cdot[K:\mathbb{Q}]}{[K(x):\mathbb{Q}(x)]} = \frac{[K(x):K]\cdot 2}{2} = [K(x):K],$$

from which the claim follows. Thus x is a zero of the polynomial

$$(X-x_1)(X-x_2)(X-x_3)(X-x_4) = (X^2 - 72954X + 729)^2$$

whose coefficients are determined by numerical approximation. Therefore we obtain

$$\sqrt{3}\frac{\varphi(3i)}{\varphi(i)} = \sqrt[24]{x} = \sqrt[24]{36477 + 21060\sqrt{3}} = \sqrt[4]{3 + 2\sqrt{3}}.$$

Example 4.11. Now, we consider $\sqrt{5}\varphi(\sqrt{5}i)/\varphi(i)$. Let $K = \mathbb{Q}\sqrt{-1})$. By Proposition 4.7 we have

$$\begin{aligned} W_{K,10} \simeq &\left\{\alpha_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \alpha_2=\begin{pmatrix}1&-4\\4&1\end{pmatrix},\ \alpha_3=\begin{pmatrix}1&-6\\6&1\end{pmatrix},\ \alpha_4=\begin{pmatrix}2&-3\\3&2\end{pmatrix},\right.\\ &\left.\alpha_5=\begin{pmatrix}2&-5\\5&2\end{pmatrix},\ \alpha_6=\begin{pmatrix}2&-7\\7&2\end{pmatrix},\ \alpha_7=\begin{pmatrix}3&0\\0&3\end{pmatrix},\ \alpha_8=\begin{pmatrix}4&-5\\5&4\end{pmatrix}\right\}.\end{aligned}$$

Since

$$x=\left(\sqrt{5}\frac{\varphi(5i)}{\varphi(i)}\right)^{120} = g_{(0,\frac{1}{5})}(i)^{120}g_{(0,\frac{2}{5})}(i)^{120}g_{(\frac{1}{2},\frac{1}{10})}(i)^{240}g_{(\frac{1}{2},\frac{3}{10})}(i)^{240} \text{ by Proposition 2.4(i) and (iii)}$$
$$\approx 414739352204549216028711957742592720 02,$$

x lies in $K_{(10)}$ by Propositions 2.2(iii) and 4.3. Hence its conjugates $x_k=x^{\alpha_k}$ $(1\leq k\leq 8)$ over K are

$$\begin{aligned}
x_1=x_5=x_7=x_8 &= g_{(0,\frac{1}{5})}(i)^{120}g_{(0,\frac{2}{5})}(i)^{120}g_{(\frac{1}{2},\frac{1}{10})}(i)^{240}g_{(\frac{1}{2},\frac{3}{10})}(i)^{240},\\
x_2 &= g_{(\frac{4}{5},\frac{1}{5})}(i)^{120}g_{(\frac{3}{5},\frac{2}{5})}(i)^{120}g_{(\frac{9}{10},\frac{1}{10})}(i)^{240}g_{(\frac{7}{10},\frac{3}{10})}(i)^{240},\\
x_3=x_6 &= g_{(\frac{1}{5},\frac{1}{5})}(i)^{120}g_{(\frac{2}{5},\frac{2}{5})}(i)^{120}g_{(\frac{1}{10},\frac{1}{10})}(i)^{240}g_{(\frac{3}{10},\frac{3}{10})}(i)^{240},\\
x_4 &= g_{(\frac{3}{5},\frac{2}{5})}(i)^{120}g_{(\frac{1}{5},\frac{4}{5})}(i)^{120}g_{(\frac{3}{10},\frac{7}{10})}(i)^{240}g_{(\frac{9}{10},\frac{1}{10})}(i)^{240}
\end{aligned}$$

with some multiplicity by Propositions 4.7 and 2.4(iv). So x is a zero of the polynomial

$$(X^2-41473935220454921602871195774259272002X+1)^4,$$

which illustrates that x is a unit. Therefore we get

$$\begin{aligned}
\sqrt{5}\frac{\varphi(5i)}{\varphi(i)} &= \sqrt[120]{x}\\
&= \sqrt[120]{20736967610227460801435597887129636001+9273853844735993106095069260699853880\sqrt{5}}\\
&= \sqrt[10]{682+305\sqrt{5}}.
\end{aligned}$$

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